Sadaa Al-qalum Journal

Neutrosophic Linear Transformations and Rank – Nullity Theorem

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ARTICLE INFO

Received: 26-04-2026

Accepted: 15-05-2026

Published: 02-06-2026

Keywords: Neutrosophic vector space, Neutrosophic linear transformations, Neutrosophic kernel, Neutrosophic Image, Neutrosophic rank, Neutrosophic nullity, Neutrosophic Rank-Nullity Theorem.

ABSTRACT

In this paper, we investigate neutrosophic linear transformation defined over finite-dimensional vector space. We rigorously formulate the concepts of the neutrosophic kernel (ker(T_I)) and neutrosophic image (Im(T_I)).establishing their underlying vector space properties. Based on these definition, we introduce the notions of neutrosophic rank and neutrosophic nullity as the dimensions of the respective subspaces over the neutrosophic field $\mathbb{R}(I)$. The primary contribution of this work is the formulation and proof of the Neutrosophic Rank-Nullity Theorem, extending the fundamental classical result of linear algebra to the neutrosophic domain: $\dim\big(V(I)\big) = \text{Rank}(T_I) + \text{Nullity}(T_I)$. Finally, several non-trivial concrete examples involving geometric transformations and neutrosophic matrix spaces are presented to illustrate the operational mechanics and validate the theoretical framework established in this study.

الملخص:

ندرس في هذه الورقة البحثية التحويلات الخطية النيوتروسوفية المعرفة على الفضاءات المتجهة النيوتروسوفية محددة الأبعاد. نقوم بصياغة مفاهيم النواة النيوتروسوفية $(\ker(T_I))$ والصورة النيوتروسوفية $(\text{Im}(T_I))$. مع إثبات خصائصهما كفضاءات فرعية. وبناءً على هذه التعاريف، نقدم مفهومي الرتبة والعدمية النيوتروسوفية كأبعاد لهذه الفضاءات الفرعية فوق الحقل النيوتروسوفي $\mathbb{R}(I)$. المساهمة الرئيسية لهذا العمل هي صياغة وإثبات نظرية الرتبة والعدمية النيوتروسوفية، لتكون امتداداً للنتيجة الكلاسيكية الأساسية في الجبر الخطي إلى المجال النيوتروسوفي: $\dim\big(V(I)\big) = \text{Rank}(T_I) + \text{Nullity}(T_I)$. وأخيراً،

نقوم بعرض أمثلة تطبيقية غير تافهة تتضمن تحويلات هندسية وفضاءات مصفوفات نيوتروسوفية لتوضيح آلية العمل وتأكيد صحة الإطار النظري المطروح.

Introduction:

Neutrosophy is a branch of philosophy introduced by Florentin Smarandache [1,2] in 1995 to study the concepts of truth, indeterminacy, and falsity. Based on this theory. Smarandache developed neutrosophic logic and neutrosophic sets as a generalization of fuzzy sets introduced by Zadeh [3] and intuitionistic fuzzy sets proposed by Atanassov [4]. These concepts provide a powerful mathematical framework for modeling uncertainty and indeterminate information arising in real—world applications.

In recent years, neutrosophic theory has attracted considerable attention and has been successfully applied in many areas of mathematics, including algebra, number theory [5,6], optimization [7], Boolean algebra [8], and algebraic equations [9,10]. The development of neutrosophic fields, which subsequently led to the study of neutrosophic vector spaces and their algebraic properties [11,12].

Neutrosophic vector spaces provide a natural extension of classical vector spaces by incorporating indeterminate components into vectors and scalars. This extension has motivated the investigation of several algebraic concepts such as neutrosophic subspaces, bases, dimension, matrices, and linear transformations [12,13,14,15]. Consequently, many classical results in linear algebra can be reformulated within the neutrosophic framework. Recently, neutrosophic vector spaces have been extended to more general settings, and several of their algebraic and topological properties have been investigated [16]. These developments have further motivated the study of neutrosophic linear transformations and their associated algebraic structures.

In this paper, we investigate neutrosophic linear transformation between neutrosophic vector spaces. We examine their fundamental properties, define the neutrosophic kernel and neutrosophic image, develop the concepts of neutrosophic rank and neutrosophic nullity, and establish several new results, including a neutrosophic version of the Rank—Nullity Theorem as an extension of the classical Rank—Nullity Theorem to neutrosophic vector spaces.

1. Preliminaries:

Definition1.1: [12] Let $(K,+,\cdot)$ be any field, and let I denote the neutrosophic indeterminacy factor. The neutrosophic field generated by K and I, denoted by $(K(I),+,\cdot)$ is defined as:

$$K(I) = \langle K \cup I \rangle = \{a + bI : a, b \in K, I^2 = I\}$$

Where:

- The additive identity $0 \in K$ is represented as $0 + 0I \in K(I)$.
- The multiplicative identity $1 \in K$ is represented as $1 + 0I \in K(I)$.

Definition1.2: [18,19] Let $(V, +, \cdot)$ be any vector space over a field K, let I denote the neutrosophic indeterminacy factor. The neutrosophic vector space generated by V and I over the field $K(I)$, denoted by $(V(I), +, \cdot)$ is defined as:

$$V(I) = \langle V \cup I \rangle = \{u + vI : u, v \in V, I^2 = I\}$$

Where:

1) $u + v \in V(I) \ \forall\, u, v \in V(I)$.
2) $\alpha u \in V(I) \ \forall \alpha \in K(I), u \in V(I)$.
3) $u + v = v + u \ \forall\, u, v \in V(I)$.
4) $(u + v) + w = u + (v + w) \ \forall\, u, v, w \in V(I)$.
5) $\exists 0 \in V(I) \ni u + 0 = 0 + u = u \in V(I)$.
6) $\forall\, u \in V(I)\ \exists (-u) \in V(I) \ni u + (-u) = 0$.
7) $\alpha(u + v) = \alpha u + \alpha v \ \forall\, \alpha \in K(I), u, v \in V(I)$.
8) $(\alpha + \beta)u = \alpha u + \beta u \ \forall\, \alpha, \beta \in K(I), u \in V(I)$.
9) $(\alpha\beta)u = \alpha(\beta u) \ \forall\, \alpha, \beta \in K(I), u \in V(I)$.
10) $1u = u \ \forall 1 \in K(I), u \in V(I)$.

Remark:[12] The elements of $V(I)$ are called neutrosophic vectors and the elements of $K(I)$ are called neutrosophic scalars.

Definition1.3:[12,19] Let $V(I)$ be a neutrosophic vector space over the neutrosophic field $K(I)$. A nonempty subset $W(I)$ of $V(I)$ is called a neutrosophic subspace of $V(I)$ if $W(I)$ itself forms a neutrosophic vector space under the vector addition and scalar multiplication inherited from $V(I)$. Equivalently, for every $u, v \in W(I)$ and every scalar $\alpha \in K(I)$, the following conditions hold:

1) $u + v \in W(I)$.
2) $\alpha u \in W(I)$.

Hence, $W(I)$ is closed under vector addition and scalar multiplication.

Definition1.4: [12,13] Let $V(I)$ be a neutrosophic vector space over a neutrosophic field $K(I)$. A neutrosophic vector $v \in V(I)$ is said to be a neutrosophic linear combination of the neutrosophic vectors $v_1, v_2, \dots, v_n \in V(I)$ if there exist scalars $(\alpha_1, \alpha_2, \dots, \alpha_n \in K(I))$ such that:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the neutrosophic coefficients of the linear combination.

Definition1.5: [12,14] Let $V(I)$ be a neutrosophic vector space over a neutrosophic field $K(I)$, $S = \{v_1, v_2, \dots v_n\}$ be a subset of neutrosophic vectors in $V(I)$. The neutrosophic span of S, denoted by $N-$ span(S) or S_N, is the set of all neutrosophic linear combination of the vectors in S.

Definition1.6: [13,14] Let $V(I)$ be a neutrosophic vector space over a neutrosophic field $K(I)$. A finite set of neutrosophic vectors $v_1, v_2, \dots, v_n$in a neutrosophic vector space $V(I)$ is said to be neutrosophically linearly independent if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 + 0I$$

Implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, where $\alpha_1, \alpha_2, \dots, \alpha_n \in K(I)$.

If there exists at least one non- trivial solution (where not all α_i are 0+0I), the set of vectors is said to be neutrosophically linearly dependent.

Definition1.7: [13,14] Let $V(I)$ be a neutrosophic vector space over a neutrosophic field $K(I)$, let $S_N = \{v_1, v_2, \dots, v_n\}$ be a finite set of neutrosophic vectors in $V(I)$. Then, S_N is called a neutrosophic basis for the neutrosophic vector space $V(I)$ if the following two conditions are satisfied:

1) S_N is neutrosophically linearly independent.
2) S_N neutrosophically spans $V(I)$.

Definition 1.8: [14] Let $V(I)$ be a neutrosophic vector space over a neutrosophic field $K(I)$. If $V(I)$ has a finite neutrosophic basis $S_N = \{v_1, v_2, \dots, v_n\}$, then the neutrosophic dimension of $V(I)$, denoted by

$\dim(V(I))$, is defined as the number of vectors in the neutrosophic basis. That is, $\dim(V(I)) = |S_N| = n.$ If no finite neutrosophic basis exists, then $V(I)$ is said to be infinite – dimensional.

2. **Neutrosophic Linear Transformations:**

Definition 2.1: [15] Let $V_1(I)$ and $V_2(I)$ be neutrosophic vector spaces over a neutrosophic field $K(I)$, A mapping $T_I: V_1(I) \to V_2(I)$ is called a neutrosophic linear transformation if it satisfies the following two conditions for all neutrosophic vectors $u, v \in V_1(I)$ and all neutrosophic scalars $\alpha \in K(I)$:

1) $T_I(u+v) = T_I(u) + T_I(v).$
2) $T_I(\alpha u) = \alpha T_I(u).$

Where any neutrosophic vector $u = a + bI \in V_1(I)$ for all $a, b \in V$ and any neutrosophic scalar $\alpha = k + mI \in K(I)$for all $k, m \in K.$

Remark 2.2:[15] Every neutrosophic linear transformation $T_I: V_1(I) \to V_2(I)$ can be uniquely decomposed into the form $T_I = T + UI,$ where T and U are classical linear transformation.

Example2.3: Let $V(I) = \mathcal{R}^2(I) = \{(u_1, u_2): u_1, u_2 \in \mathcal{R}(I)\}$ be a neutrosophic vector space over the neutrosophic field $K(I) = \mathcal{R}(I).$ Consider the mapping $T_I: V(I) \to V(I)$ defined by:

$T_I(u) = T_I(u_1, u_2) = (u_1 + u_2, u_1 - u_2)$ for all $u = (u_1, u_2) \in V(I)$

Show that T_I is a neutrosophic linear transformation.

Let $u = (a + bI, c + dI), v = (e + fI, g + hI), \alpha = (k + mI)$ then:

1) $T_I(u+v) = T_I\big((a+bI, c+dI) + (e+fI, g+hI)\big).$

$$= T_I\big((a+e) + (b+f)I, (c+g) + (d+h)I\big).$$
$$= ((a+e+c+g) + (b+f+d+h)I,$$
$$(a+e-c-g) + (b+f+d+h)I).$$
$$= \big((a+c) + (b+d)I, (a-c) + (b-d)I\big)$$
$$+ \big((e+g) + (f+h)I, (e-g) + (f-h)I\big).$$
$$= T_I(u) + T_I(v).$$

2) $T_I(\alpha u) = T_I\big((k+mI)(a+bI), (k+mI)(c+dI)\big)$

$$= T_I(ka + [kb + ma + mb]I, kc + [kd + mc + md]I).$$
$$= (k(a + c) + [kb + kd + ma + mc + mb + md]I,$$
$$k(a - c) + [kb - kd + ma - mc + mb - md]I).$$
$$= (k + mI)((a + c + (b + d)I, (a - c) + (b - d)I).$$
$$= \alpha T_I(u).$$

Hence, T_I is a neutrosophic linear transformation.

Definition 2.4: Let $V_1(I)$ and $V_2(I)$ be two neutrosophic vector spaces over a neutrosophic field $K(I)$, and let $T_I: V_1(I) \to V_2(I)$ be a neutrosophic linear transformation. The neutrosophic kernel of T_I denoted by $\ker(T_I)$ is defined as:

$$\ker(T_I) = \{v \in V_1(I): T_I(v) = 0_{V_2(I)} = 0 + 0I\}$$

And the neutrosophic Image of T_I denoted by $\mathrm{Im}(T_I)$ is defined as:

$$\mathrm{Im}(T_I) = \{T_I\ (x + yI) \in V_2(I): x + yI \in V_1\ (I)\}$$

Definition 2.5: Let $V_1(I)$ and $V_2(I)$ be two neutrosophic vector spaces over a neutrosophic field $K(I)$, and let $T_I: V_1(I) \to V_2(I)$ be a neutrosophic linear transformation. The dimension of the neutrosophic kernel space is called neutrosophic nullity, denoted by:

$$\mathrm{Nullity}(T_I) = \dim(\ker\ (T_I)).$$

And the dimension of the neutrosophic image space is called neutrosophic rank denoted by:

$$\mathrm{Rank}(T_I) = \dim\ (\mathrm{Im}(T_I)).$$

3. Main Results:

Lemma 3.1: Let $T_I: V_1(I) \to V_2(I)$ be a neutrosophic linear transformation, where $V_1(I)$ and $V_2(I)$ are neutrosophic vector spaces over a neutrosophic field $K(I)$. Then $T_I(0_{V_{1(I)}}) = 0_{V_{2(I)}}$.

Proof:

Let $u \in V_1(I)$ be any neutrosophic vector, since $0_{V_{1(I)}}$ is the zero vector of $V_1(I)$, we have $u + 0_{V_{1(I)}} = u$. Applying the transformation T_I to both sides, we get $T_I\left(u + 0_{V_{1(I)}}\right) = T_I(u)$. Since T_I is a neutrosophic linear transformation, $T_I(u) + T_I(0_{V_1(I)}) = T_I(u)$. Adding the additive inverse $-T_I(u) \in V_2(I)$ to both sides of the equation $\left(T_I(u) + T_I\left(0_{V_{1(I)}}\right)\right) - T_I(u) = T_I(u) - T_I(u)$. Hence $T_I\left(0_{V_{1(I)}}\right) = 0_{V_{2(I)}}$.

Theorem 3.2: Let $T_I: V_1(I) \to V_2(I)$ be a neutrosophic linear transformation. Then, the kernel of T_I is a neutrosophic subspace of V_1(I).

Proof:

To prove that ker(T_I) is a neutrosophic subspace of $V_1(I)$, we must verify the following three condition:-

1) Non – emptiness:

By lemma (3.1), we have $T_I\left(0_{V_{1(I)}}\right) = 0_{V_{2(I)}}$. Thus, $0_{V_{(I)}} \in$ ker(T_I). Therefore ker $T_I \neq \emptyset$.

2) Closure under neutrosophic vector addition:

Let $u, v \in$ ker(T_I). Then, by definition $T_I(u) = 0_{V_{2(I)}}$ and $T_I(V) = 0_{V_{2(I)}}$. Since T_I is a neutrosophic vector addition, $T_I(u+v) = T_I(u) + T_I(V) = 0_{V_{2(I)}} + 0_{V_{2(I)}} = 0_{V_{2(I)}}$. Therefore, $(u+v) \in$ ker(T_I).

3) Closure under neutrosophic scalar multiplication:

Let $u \in$ ker(T_I) and let $\alpha \in$ ker(I) be any neutrosophic scalar. Since $T_I(u) = 0_{V_{2(I)}}$, we have $T_I(\alpha u) = \alpha T_I(u) = \alpha 0_{V_{2(I)}} = 0_{V_{2(I)}}$, Therefore, $(\alpha u) \in$ ker(T_I).

Since Ker(T_I) is non – empty and closed under both neutrosophic vector addition and scalar multiplication. It satisfies all the conditions of neutrosophic subspace. Therefore Ker(T_I) is a neutrosophic subspace of $V_1(I)$.

Theorem 3.3: Let $T_I: V_1(I) \to V_2(I)$ be a neutrosophic linear transformation. Then, the image of T_I is a neutrosophic subspace of $V_2(I)$.

Proof:

To prove that Im(T_I) is a neutrosophic subspace of $V_2(I)$, we must verify the following three conditions:-

1) Non – emptiness:

By lemma (3.1), we have $T_I(0_{V_{1(I)}}) = 0_{V_{2(I)}}$. Since $0_{V_{1(I)}} \in V_1(I)$, it follows that $0_{V_{2(I)}} \in \text{Im}(T_I)$. Thus, $\text{Im}(T_I) \neq \emptyset$.

2) Closure under neutrosophic vector addition:

Let $w_1, w_2 \in \text{Im}(T_I)$. By definition, there exist $u_1, u_2 \in V_1(I)$ such that $T_I(u_1) = w_1$ and $T_I(u_2) = w_2$. Since $V_1(I)$ is a neutrosophic vector space, $(u_1 + u_2) \in V_1(I)$. Using the additivity property of T_I, we get $w_1 + w_2 = T_I(u_1) + T_I(u_2) = T_I(u_1 + u_2)$. Since $w_1 + w_2$ is the image of $u_1 + u_2 \in V_1(I)$, it follows that $(w_1 + w_2) \in \text{Im}(T_I)$.

3) Closure under neutrosophic scalar multiplication:

Let $w \in \text{Im}(T_I)$ and let $\alpha \in K(I)$ be any neutrosophic scalar. Since $w \in \text{Im}(T_I)$, there exists $u \in V_1(I)$ such that $T_I(u) = w$. Since $V_1(I)$ is closed under scalar multiplication, $\alpha u \in V_1(I)$, we have $\alpha w = \alpha T_I(u) = T_I(\alpha u)$. Since αw is the image of $\alpha u \in V_1(I)$, it follows that $\alpha w \in \text{Im}(T_I)$.

Since Im (T_I) is non – empty and closed under both neutrosophic vector addition and scalar multiplication, $\text{Im}(T_I)$ is a neutrosophic subspace of $V_2(I)$.

Theorem 3.4: Let $T_I: V(I) \to W(I)$ and $S_I: W(I) \to U(I)$ be two neutrosophic linear transformations over $K(I)$. Then the composite mapping $(S_I \circ T_I): V(I) \longrightarrow U(I)$ defined by: $(S_I \circ T_I)(v) = S_I(T_I(v))$ for all $v \in V(I)$ is also a neutrosophic linear transformation.

Proof:

If $u, v \in V(I), \alpha \in K(I)$ then:

1) $$(S_I \circ T_I)(u + v) = S_I\big(T_I(u + v)\big)$$
$$= S_I(T_I(u) + T_I(v))$$
$$= S_I\big(T_I(u)\big) + S_I(T_I(v))$$
$$= (S_I \circ T_I)(u) + (S_I \circ T_I)(v).$$

2) $$(S_I \circ T_I)(\alpha u) = S_I\big(T_I(\alpha u)\big).$$
$$= S_I(\alpha T_I(u))$$
$$= \alpha S_I(T_I(u))$$
$$= \alpha(S_I \circ T_I)(u).$$

Since $(S_I \circ T_I)$ preserves both neutrosophic vector addition and scalar multiplication, it follows that $(S_I \circ T_I)$ is a neutrosophic linear transformation from $V(I)$ to $U(I)$.

Theorem 3.5: Let $T_I, S_I: V(I) \rightarrow W(I)$ be two neutrosophic linear transformations over $K(I)$. Then the sum $(T_I + S_I): V(I) \rightarrow W(I)$ defined by:

$(T_I + S_I)(v) = T_I(v) + S_I(v)$ for all $v \in V(I)$ is also a neutrosophic linear transformation.

Proof:

If $u, v \in V(I), \alpha \in K(I)$ then:

1) $$(T_I + S_I)(u + v) = T_I(u + v) + S_I(u + v)$$
$$= \big(T_I(u) + T_I(v)\big) + (S_I(u) + S_I(v))$$
$$= \big(T_I(u) + S_I(u)\big) + (T_I(v) + S_I(v))$$
$$= (T_I + S_I)(u) + (T_I + S_I)(v).$$

2) $$(T_I + S_I)(\alpha u) = T_I(\alpha u) + S_I(\alpha u)$$
$$= \alpha T_I(u) + \alpha S_I(u)$$
$$= \alpha(T_I(u) + S_I(u))$$
$$= \alpha(T_I + S_I)(u).$$

Since $(T_I + S_I)$ satisfies both additivity and scalar homogeneity, it is a neutrosophic linear transformation from $V(I)$ to $w(I)$.

Theorem 3.6: Let $T_I: V(I) \rightarrow W(I)$ be a neutrosophic linear transformation over $K(I)$, and let $\alpha \in K(I)$ be any neutrosophic scalar. Then the scalar product $(\alpha T_I): V(I) \rightarrow W(I)$ defined by:

$(\alpha T_I)(v) = \alpha T_I(v)$ for all $v \in V(I)$ is also a neutrosophic linear transformation.

Proof:

If $u, v \in V(I), \beta \in K(I)$ then:

1) $$(\alpha T_I)(u + v) = \alpha T_I(u + v)$$
$$= \alpha\big(T_I(u) + T_I(v)\big)$$
$$= \alpha T_I(u) + \alpha T_I(v)$$

$$= (\alpha T_I)(u) + (\alpha T_I)(v).$$

2) $(\alpha T_I)(\beta v) = \alpha T_I(\beta v)$

$$= \alpha(\beta T_I(v))$$
$$= (\alpha\beta) T_I(v)$$
$$= (\beta\alpha) T_I(v)$$
$$= \beta(\alpha T_I(v))$$
$$= \beta(\alpha T_I)(v).$$

Since (αT_I) satisfies both additivity and scalar homogeneity, (αT_I) is a neutrosophic linear transformation from $V(I)$ to $W(I)$.

Theorem 3.7: Let $V(I)$ and $W(I)$ be finite- dimensional neutrosophic vector spaces over a neutrosophic field $K(I)$, and let $T_I: V(I) \to W(I)$ be a neutrosophic linear transformation. Then:

dim$(V(I))$ = Rank(T_I) + Nullity(T_I).

Proof:

Let Nullity$(T_I) = m$ and dim$\big(V(I)\big) = n$. Since ker(T_I) is a neutrosophic subspace of $V(I)$ (by Theorem 3.2), it has a finite basis. Let $S_{N,ker} = \{v_1, v_2, \dots, v_m\}$ be a basis for ker(T_I). By the Basis Extension Theorem for neutrosophic vector spaces, we can extend $S_{N,ker}$ to a neutrosophic basis for the entire space $V(I)$, say $B_N = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$. To prove the theorem, it suffices to show that the set $S_N = \{T_I(v_{m+1}), T_I(v_{m+2}), \dots, T_I(v_n)\}$ is a neutrosophic basis for Im(T_I), which would imply that:

Rank(T_I) =dim(ImT_I)) $= n - m =$ dim$\big(V(I)\big) -$ Nullity(T_I). We verify the two conditions for S_N being a neutrosophic basis of Im(T_I): $-$

1) S_N spans Im(T_I)(span $(S_N) = Im(T_I)$):

Let $w \in Im(T_I)$. Then there exists a vector $v \in V(I)$ such that $T_I(v) = w$. Since B_N is a neutrosophic basis for $V(I)$, we can express v uniquely as a neutrosophic linear combination over $K(I)$, $v = \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_n v_n$, where $\alpha_i \in K(I)$ for all $i = 1, \dots, n$. Applying T_I and using its linearity,

$$w = T_I(v) = T_I(\alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \cdots + \alpha_n v_n)$$

$$= \alpha_1 T_I(v_1) + \cdots + \alpha_m T_I(v_m) + \alpha_{m+1} T_I(v_{m+1}) + \cdots + \alpha_n T_I(v_n).$$

Since $v_1, \dots, v_m \in \ker(T_I)$, we have $T_I(v_1) = \cdots = T_I(v_m) = 0_{W(I)}$. Thus, the equation simplifies to $w = \alpha_{m+1} T_I(v_{m+1}) + \cdots + \alpha_n T_I(v_n)$. Hence every vector $w \in \mathrm{Im}(T_I)$ is a neutrosophic linear combination of elements in S_N, so span $(S_N) = \mathrm{Im}(T_I)$.

2) S_N is neutrosophically linearly independent over $K(I)$:

Suppose there exist neutrosophic scalars $\beta_{m+1}, \dots, \beta_n \in K(I)$ such that, $\beta_{m+1} T_I(v_{m+1}) + \cdots + \beta_n T_I(v_n) = 0_{W(I)}$. By the linearity of T_I, this can be written as, $T_I(\beta_{m+1} v_{m+1} + \cdots + \beta_n v_n) = 0_{W(I)}$. This implies that the vector $(\beta_{m+1} v_{m+1} + \cdots + \beta_n v_n) \in \ker(T_I)$. Since $S_{N,Ker} = \{v_1, \dots, v_m\}$ spans ker (T_I), there exist neutrosophic scalars $c_1, \dots, c_m \in K(I)$ such that, $\beta_{m+1} v_{m+1} + \cdots + \beta_n v_n = c_1 v_1 + \cdots + c_m v_m$. Rearranging the terms, $c_1 v_1 + \cdots + c_m v_m - \beta_{m+1} v_{m+1} - \cdots - \beta_n v_n = 0_{V(I)}$. Since $\{v_1, \dots, v_n\}$ is neutrosophically linearly independent in $V(I)$, all coefficients must be zero, which implies $c_1 = \cdots = c_m = \beta_{m+1} = \cdots = \beta_n = 0_{K(I)}$. Therefore, S_N is neutrosophically linearly independent over $K(I)$. Since S_N spans $\mathrm{Im}(T_I)$ and is neutrosophic linearly independent, S_N forms a neutrosophic basis for $\mathrm{Im}(T_I)$. Consequently, dim $(\mathrm{Im}(T_I)) = |S_N| = n - m$. Since dim $(\ker(T_I)) = m$ and $\dim\big(V(I)\big) = n$, we conclude that, dim $(\ker(T_I))$ + dim $(\mathrm{Im}(T_I)) = m + (n - m) = n = \dim(V(I))$. Hence, $\dim\big(V(I)\big) = \mathrm{Rank}(T_I) + \mathrm{Nullity}(T_I)$.

Example 3.8: Consider the neutrosophic linear transformation $T_I: \mathcal{R}^2(I) \to \mathcal{R}^2(I)$ defined in Example (2.3) by:

$T_I(u_1, u_2) = (u_1 + u_2, u_1 - u_2)$ for all $(u_1, u_2) \in \mathcal{R}^2(I)$.

Find $\ker(T_I)$ and $\mathrm{Im}(T_I)$.

1) To find $\ker(T_I)$:

Let $u = (a + bI, c + dI) \in \ker(T_I)$, where $a, b, c, d \in \mathcal{R}$. By the definition of the kernel:

$$T_I(u) = (0 + 0I, 0 + 0I)$$

Applying the neutrosophic transformation T_I:

$$\big((a+bI)+(c+dI),(a+bI)-(c+dI)\big)=(0+0I,0+0I)$$

$$\Longrightarrow \big((a+c)+(b+d)I,(a-c)+(b-d)I\big)=(0+0I,0+0I)$$

Equating the respective components, we obtain the system of equations:

$$\begin{cases} a+c=0 \\ a-c=0 \end{cases} \text{ and } \begin{cases} b+d=0 \\ b-d=0 \end{cases}$$

$$\Longrightarrow a=b=c=d=0$$

Thus, $u=(0+0I,0+0I)$.

Therefore,

$$\ker(T_I)=\{(0+0I,0+0I)\}.$$

2) To fine $\mathrm{Im}(T_I)$:

Let $w=(x_1+y_1I,x_2+y_2I)\in\mathcal{R}^2(I)$ be an arbitrary vector. We search for a vector $u=(a+bI,c+dI)\in\mathcal{R}^2(I)$ such that $T_I(u)=w$. This leads to:

$$\big((a+c)+(b+d)I,(a-c)+(b-d)I\big)=(x_1+y_1I,x_2+y_2I).$$

By equating the real and neutrosophic parts:

$$\begin{cases} a+c=x_1 \\ a-c=x_2 \end{cases} \Longrightarrow\ a=\frac{x_1+x_2}{2},c=\frac{x_1-x_2}{2}$$

$$\begin{cases} b+d=y_1 \\ b-d=y_2 \end{cases} \Longrightarrow b=\frac{y_1+y_2}{2},d=\frac{y_1-y_2}{2}$$

Since $a,b,c,d\in\mathcal{R}$ exist for any choice of $x_1,y_1,x_2,y_2\in\mathcal{R}$, every neutrosophic vector $w\in\mathcal{R}^2(I)$ has a pre-image under $T_{I.}$

Therefore, $\mathrm{Im}(T_I)=\mathcal{R}^2(I)$.

Example 3.9: Let $V(I)=\mathcal{R}^2(I)=\{(u_1,u_2):u_1,u_2\in\mathcal{R}(I)\}$ be a neutrosophic vector space over $\mathcal{R}(I)$. Define the neutrosophic linear transformation $T_I:V(I)\to V(I)$ by:

$T_I(u)=T_I(u_1,u_2)=(-u_2,u_1)$ for all $u=(u_1,u_2)\in V(I)$

Show that the Rank-Nullity Theorem holds for T_I.

1) To find $\ker(T_I)$:

Let $u = (a_1 + b_1 I, a_2 + b_2 I) \in \ker(T_I)$, where $a_1, b_1, a_2, b_2 \in \mathcal{R}$. By the definition of the kernel,

$$T_I(u) = (0 + 0I, 0 + 0I)$$

$$T_I(a_1 + b_1 I, a_2 + b_2 I) = (-(a_2 + b_2 I), a_1 + b_1 I) = (0 + 0I, 0 + 0I)$$

Equating the respective components:

$$\begin{cases} -a_2 - b_2 I = 0 + 0I \Longrightarrow a_2 = 0, b_2 = 0 \Longrightarrow u_2 = 0 + 0I \\ a_1 + b_1 I = 0 + 0I \Longrightarrow a_1 = 0, b_1 = 0 \Longrightarrow u_1 = 0 + 0I \end{cases}$$

Therefore, $\ker(T_I) = \{(0 + 0I, 0 + 0I)\} = \{0_{V(I)}\}$.

Hence, $\text{Nullity}(T_I) = \dim(\ker(T_I)) = 0$.

2) To find $\text{Im}(T_I)$:

Let $w = (w_1, w_2) = (c_1 + d_1 I, c_2 + d_2 I) \in \mathcal{R}^2(I)$ be an arbitrary vector, where $c_1, d_1, c_2, d_2 \in \mathcal{R}$. We seek a vector $u = (a_1 + b_1 I, a_2 + b_2 I) \in V(I)$ such that $T_I(u) = w$:

$$(-(a_2 + b_2 I), a_1 + b_1 I) = (c_1 + d_1 I, c_2 + d_2 I)$$

Equating the components gives:

$$\begin{cases} -a_2 - b_2 I = c_1 + d_1 I \Longrightarrow a_2 = -c_1, b_2 = -d_1 \\ a_1 + b_1 I = c_2 + d_2 I \Longrightarrow a_1 = c_2, b_1 = d_2 \end{cases}$$

Therefore, $\text{Im}(T_I) = \mathcal{R}^2(I)$.

Hence, $\text{Rank}(T_I) = \dim(\text{Im}(T_I)) = 2$.

3) Verification of the Rank-Nullity Theorem:

Since $\dim\big(V(I)\big) = \dim\big(\mathcal{R}^2(I)\big) = 2$, we have:

$\text{Rank}(T_I) + \text{Nullity}(T_I) = 2 + 0 = 2 = \dim\big(V(I)\big)$.

Thus, the Rank-Nullity Theorem holds.

Example 3.10: Let $V(I) = \left\{\begin{bmatrix} A & 0 + 0I \\ 0 + 0I & 0 + 0I \end{bmatrix} : A \in \mathcal{R}(I)\right\} \subset M_{2\times 2}$ be a neutrosophic subspace of $M_{2\times 2}$. Define the neutrosophic linear transformation $T_I\left(\begin{bmatrix} A & 0 + 0I \\ 0 + 0I & 0 + 0I \end{bmatrix}\right) = A$

Show that the Rank-Nullity Theorem holds for T_I.

1) To find $\ker(T_I)$:

Let $M = \begin{bmatrix} A & 0+0I \\ 0+0I & 0+0I \end{bmatrix} \in \ker(T_I)$, where the component A is expressed in its explicit neutrosophic form $A = a + bI$ where $a, b \in \mathcal{R}$

By the definition of the kernel, $T_I(M) = 0 + 0I$.

$$T_I\left(\begin{bmatrix} a+bI & 0+0I \\ 0+0I & 0+0I \end{bmatrix}\right) = a + bI = 0 + 0I$$

Equating the real and neutrosophic parts:

$a = 0$ and $b = 0 \Longrightarrow A = 0 + 0I$

Therefore, $\ker(T_I) = \left\{\begin{bmatrix} 0+0I & 0+0I \\ 0+0I & 0+0I \end{bmatrix}\right\} = \{0_{V(I)}\}$

Hence, Nullity$(T_I) = \dim(\ker(T_I)) = 0$.

2) To find $\mathrm{Im}(T_I)$:

Let $w = c + dI \in \mathcal{R}(I)$ be an arbitrary element, where $c, d \in \mathcal{R}$. We seek a matrix $M = \begin{bmatrix} a+bI & 0+0I \\ 0+0I & 0+0I \end{bmatrix} \in V(I)$ such that $T_I(M) = w$:

$$a + bI = c + dI \Longrightarrow a = c, b = d$$

Therefore, $\mathrm{Im}(T_I) = \mathcal{R}(I)$. Hence, rank$(T_I) = \dim(\mathrm{Im}(T_I)) = 1$.

3) Verification of the Rank-Nullity Theorem:

The dimension of the domain space $V(I)$ over the neutrosophic field $\mathcal{R}(I)$ is $\dim(V(I)) = 1$ (spanned by the basis matrix $\begin{bmatrix} 1+0I & 0+0I \\ 0+0I & 0+0I \end{bmatrix}$).

Checking the formula:

Rank(T_I) + Nullity$(T_I) = 1 + 0 = 1 = \dim\big(V(I)\big)$.

Thus, the Rank-Nullity Theorem holds.

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